



R-Annihilator- δ -Hollow Modules and R-Annihilator- δ -lifting Modules

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Abstract: Let M be an unitary module on associative ring with identity. A sub module N of an R-module M is called R -annihilator- δ -small ($N \ll_{a\delta} M$), if $M = N + T$ for every c-singular submodule T of an R-module M , implies that $Ann(T) = 0$, where $Ann(T) = \{r \in R : r.N = 0\}$. In this paper, we introduce the concepts of R - a - δ -hollow modules and R - a - δ -lifting modules which are generalization R - a -hollow modules and R - a -lifting modules with their properties.

Key words: c-singular, R -annihilator- δ -small, R -annihilator- a - δ -small, R - a -hollow module, R - a -lifting module, R - a - δ -hollow modules, R - a - δ -lifting modules.

1. Introduction:

Throughout this paper R is an associative ring with unity module and all modules are unitary left R -modules. A sub module K of a module M , $K \leq M$. Let M be a module, $K \leq M$ is said to be small in M if for every $L \leq M$, the equality $K + L = M$ implies $L = M$, (denoted by $K \ll M$). The singular of an R -module M is the set $Z(M) = \{x \in M : ann(x) \subseteq_e R\}$. If $Z(M) = M$ then M is called singular module, if $Z(M) = 0$ then M is called non-singular module. Let M be an R -module. A submodule N of M is called c-singular (denoted by $N \subseteq_{cs} M$) if $\frac{M}{N}$ is a singular module. The concept of δ -small sub modules was introduced by Zhon.

A sub module K of a module M is said to be δ -small sub module of M (denoted by $K \ll_{\delta} M$) if whenever $K + L = M$, with $\frac{M}{K}$ is singular, then $L = M$. In 1975 P, Fleury introduced the concept of hollow module as follows, A module M is said to be hollow module if $M \neq 0$ and whenever $M = A + B$, where A and B are submodules of M , then either $M = A$ or $M = B$ i.e. it is hollow if $M \neq 0$ and every proper submodule of M is a small submodule of M . The concept of R -annihilator small (R - a -small) sub module was introduced by [2]. A submodule N of an R -module M is called R -annihilator small if $N + T = M$, T is submodule of M , implies that $ann_R(T) = 0$, where $Ann(T) = \{r \in R : r.T = 0\}$ and (denoted by $N \ll_a M$).

A module M is said to be hollow module if $M \neq 0$ and whenever $M = A + B$, where A and B are submodules of M , then either $M = A$ or $M = B$ i.e. it is hollow if $M \neq 0$ and every proper submodule of M is a small submodule of M . The concept of R -annihilator small (R - a -small) sub module was introduced by [2]. A submodule N of an R -module M is called R -annihilator small if $N + T = M$, T is submodule of M , implies that $ann_R(T) = 0$, where $Ann(T) = \{r \in R : r.T = 0\}$ and (denoted by $N \ll_a M$).

An R -module M is called R -annihilator-lifting module (R - a -lifting) if for any submodule N of M there exists submodules K, L of M such that $M = K \oplus L$ with $L \leq N$ and $N \cap K \ll_a N$ with $ann(N) = 0$.

[4] A submodule N of an R-module M is called R -annihilator- δ -small ($N \ll_{a\delta} M$), if $M = N + T$ for every c-singular submodule T of an R-module M , implies that $Ann(T) = 0$. An R -module M is called lifting module if for any submodule N of M there exists submodules K, L of M such that $M = K + L$ with $L \leq N$ and $N \cap K \ll N$. An R -module M is called δ -lifting module if for any submodule N of M there exists submodules K, L of M such that $M = K + L$ with $L \leq_{cs} N$ and $N \cap K \ll N$. An R -module M is



called R -annihilator- δ -lifting module (R - a - δ -lifting) if for any submodule N of M there exists submodules K, L of M such that $M = K \oplus L$ with $L \leq_{cs} N$ and $N \cap K \ll_{a\delta} N$ with $\text{ann}(N) = 0$.

2. R -annihilator small:

Remarks: [2].

1. Let A and B be submodules of an R -module M such that $A \subseteq B$. If $A \ll_{a\delta} B$, then $A \ll_{a\delta} M$.
2. Let A and B be submodules of an R -module M such that $A \subseteq B$. If $B \ll_{a\delta} M$, then $A \ll_{a\delta} M$.
3. Let M and N be R -modules and $f: M \rightarrow N$ be epimorphism and $B \subseteq N$. If $B \ll_{a\delta} N$, then $f(B) \ll_{a\delta} M$.
4. Let $M = M_1 \oplus M_2$. If N_1 and N_2 are R -annihilator- δ -small submodules of M_1 and M_2 respectively, thus $N_1 \oplus N_2 \ll_{a\delta} M_1 \oplus M_2$.

Proof: 1). Let $M = A + X$, where X is a submodule of M . By modular law $M = A + B = A + (X \cap B)$. Since A is a R - a -small of B , then $\text{ann}(X \cap B) = 0$. Now $(X \cap B) \subseteq X \Rightarrow \text{ann}(X \cap B) \supseteq \text{ann}(X)$ for this $r \in \text{ann}(X) \Rightarrow r.X = 0$. Since $(X \cap B) \subseteq X$ therefore $r.(X \cap B) \Rightarrow r \in \text{ann}(A \cap X)$. So $\text{ann}(X) = 0$, thus A is a R - a -small of M .

2). Obvious

3). Let M and N be R -modules and $f: M \rightarrow N$ be epimorphism and $B \subseteq N$. Let B is a R - a -small of N . To prove that $f^{-1}(B)$ is a R - a -small of M . Let $M = f^{-1}(B) + X$, where X is a submodule of M . Then $f(M) = f\{f^{-1}(B) + X\} \Rightarrow f(M) = f(f^{-1}(B) + f(X))$. Since f is epic. Therefore $N = B + f(X)$. Since B is a R - a -small of N i.e. $\text{ann}f(X) = 0$. Now claim that $\text{ann}X \subseteq \text{ann}f(X)$, for this let $r \in \text{ann}(X) \Rightarrow r.X = 0$. Since $f(X) \subseteq X$, therefore $r.f(X) = f(r.X) = f\{0\} = 0$. thus $r \in f(X)$ means $\text{ann}X \subseteq \text{ann}f(X)$. Thus $\text{ann}(X) = 0$. hence $f^{-1}(B)$ is a R - a -small of M .

4). Let $N \ll_{a\delta} M$ and the projection map $\rho_1: M_1 \oplus M_2 \rightarrow M_1$. Since $N_1 \ll_{a\delta} M_1$, then we have $\rho_1^{-1}(N_1) = N_1 \oplus M_2$ is a R - a - δ -small of $M_1 \oplus M_2$. Since $N_1 \oplus N_2 \subseteq M_1 \oplus M_2$, therefore $N_1 \oplus N_2 \ll_{a\delta} M_1 \oplus M_2$.

Proposition:2.1 Let K and N be submodules of an R -modules of M with $K \subseteq_{cs} N$. and $\frac{N}{K} \ll_{a\delta} \frac{M}{K}$. Let

$\pi: M \rightarrow \frac{M}{K}$ be a natural mapping therefore $\pi^{-1}(\frac{M}{K}) \ll_{a\delta} M$, but $\pi^{-1}(\frac{N}{K}) = N$, then $N \ll_{a\delta} M$.

Proof: Obvious;

Proposition:2.2 Let M be an R -module and $A \subseteq X \subseteq B \subseteq M$ such that $\frac{B}{X} \ll_{a\delta} \frac{M}{X}$, then $\frac{B}{A} \ll_{a\delta} \frac{M}{A}$.

Proof: Consider the map $f: \frac{M}{A} \rightarrow \frac{M}{X}$ defined by $f(m+A) = m+X$ for every $m \in M$, So f is epic. Since $\frac{B}{X} \ll_{a\delta} \frac{M}{X}, \frac{B}{X}$ therefore $\frac{B}{A} = f^{-1}(\frac{B}{X}) \ll_{a\delta} \frac{M}{A}$. Thus $\frac{B}{A} \ll_{a\delta} \frac{M}{A}$.

Corollary:2.3 Let M be an R -module and let $K_1 \subseteq M_1 \subseteq M$ and $K_2 \subseteq M_2 \subseteq M$. If $\frac{M_1 + M_2}{K_1 + K_2} \ll_{a\delta} \frac{M}{K_1 + K_2}$, Then

1. $\frac{M_1 + K_2}{K_1} \ll_{a\delta} \frac{M}{K_1}$.
2. $\frac{K_1 + M_2}{K_2} \ll_{a\delta} \frac{M}{K_2}$.
3. $\frac{M_1}{K_1} \oplus \frac{M_2}{K_2} \ll_{a\delta} \frac{M}{K_1} \oplus \frac{M}{K_2}$.

Proof: 1. Consider the maps $\pi_1: \frac{M}{K_1} \rightarrow \frac{M}{K_1 + K_2}$ defined by $\pi_1(m + K_1) = (m + K_1 + K_2) \forall m \in M$ and

$\pi_2: \frac{M}{K_2} \rightarrow \frac{M}{K_1 + K_2}$ defined by $\pi_2(m + K_2) = (m + K_1 + K_2), \forall m \in M$. So π_1 and π_2 are

epimorphisms. Therefore $\frac{M_1 + K_2}{K_1 + K_2} \subseteq \frac{M_1 + M_2}{K_1 + K_2}$. Since $\frac{M_1 + M_2}{K_1 + K_2} \ll_{a\delta} \frac{M}{K_1 + K_2}$, therefore $\frac{M_1 + K_2}{K_1 + K_2}$

$\ll_{a\delta} \frac{M}{K_1 + K_2}$. We have by previous prop. $\frac{M_1 + K_2}{K_1} = \pi_1^{-1}(\frac{M_1 + M_2}{K_1 + K_2}) \ll_{a\delta} \frac{M}{K_1}$. ..1

2. Since $\frac{K_1 + M_2}{K_1 + K_2} \subseteq \frac{M_1 + M_2}{K_1 + K_2}$ and $\frac{M_1 + M_2}{K_1 + K_2} \ll_{a\delta} \frac{M}{K_1 + K_2}$, therefore $\frac{K_1 + M_2}{K_1 + K_2} \ll_{a\delta} \frac{M}{K_1 + K_2}$.

We have by previous prop. $\frac{M_1 + M_2}{K_2} = \pi_2^{-1}(\frac{K_1 + M_2}{K_1 + K_2}) \ll_{a\delta} \frac{M}{K_2}$. ..2

3. From equation (1) and (2), we obtain $\frac{M_1}{K_1} \oplus \frac{M_2}{K_2} \ll_{a\delta} \frac{M}{K_1} \oplus \frac{M}{K_2}$.

Remark: 1. An R -module M is R - a -lifting module if and only if for any sub module N of M there exists a submodule L of M such that $M = L \oplus K$ and $L \cap K \ll_a M$.

2. Every R - a -hollow module is R - a -lifting module.



3. R-Annihilator δ -Hollow Modules:

Examples and Remarks:

1. Consider Z as Z -module is $R-a-\delta$ -hollow module but it is not hollow.
2. Consider Z_4 and Z_6 as Z -module are not $R-a-\delta$ -hollow module.
3. If M be a faithful R -module, then $R-a-\delta$ -hollow module and hollow are equivalent.

Proposition:3.1 Let M be a faithful R -module and $\text{ann}(N) \leq_{cs} N \subseteq M$, then M is $R-a-\delta$ -hollow module.

Proof: [2].

The epimorphic image of $R-a-\delta$ -hollow module need not be $R-a-\delta$ -hollow module as the follow example. Consider Z_4 and Z as Z -module and let $\pi: Z \rightarrow Z_4$ be canonical epimorphism. Z as Z -module is $R-a-\delta$ -hollow module, $\{0\} \leq_{a\delta} Z$. But $\pi\{0\} = 0$ is not $R-a-\delta$ -small submodule in Z_4 where $Z_4 + 0 + Z_4$ and $\text{ann}(Z_4) = 4Z \neq 0$.

Proposition:3.2 Let M and N be R -modules and $f: M \rightarrow N$ be an epimorphism. If N is $R-a-\delta$ -hollow module, Then M is $R-a-\delta$ -hollow module.

Proof: Assume that $f: M \rightarrow N$ be an epimorphism. Let K be a submodule of M , then $f(K)$ is submodule of N and since N is $R-a-\delta$ -hollow module, therefore $f(K)$ is also $R-a-\delta$ -hollow module, then $f^{-1}(f(K)) \leq_{as} M$ by remark 1.3, $f^{-1}(f(K)) = K + \ker f$. Now $K \leq K + \ker f \Rightarrow \ker f = 0$ i.e. $K \leq_{a\delta} M$. Thus M is $R-a-\delta$ -hollow module.

Corollary:3.3 Let M be an R -module and D be c-singular submodule of M . $\frac{M}{D}$ is $R-a-\delta$ -hollow module with $\text{ann}(D) = 0$, then M is $R-a-\delta$ -hollow module with $\text{ann}(D) = 0$.

A submodule N of an R -module M is called fully invariant submodule of M if $f(N) \subseteq N$ for every $f \in \text{End}_R(M)$. [5] A module M is called duo-module if every submodule of M is fully invariant. [8].

Proposition:3.4 Let $M = D_1 \oplus D_2$ be duo-module. If D_1 and D_2 are $R-a-\delta$ -hollow modules, then M is $R-a-\delta$ -hollow module.

Proof: Let D_1 and D_2 are $R-a-\delta$ -hollow modules and $N_1 \oplus N_2$ be a c-singular submodule of $D_1 \oplus D_2$, where $N_1 \leq_{cs} D_1$ and $N_2 \leq_{cs} D_2$, then N_1 and N_2 are $R-a-\delta$ -small submodules of D_1 and D_2 respectively, thus by remarks 4 $N_1 \oplus N_2$ is $R-a-\delta$ -small submodules of M . Thus M is $R-a-\delta$ -hollow module.

Corollary:3.5 Let $M = D_1 \oplus D_2$ be an R -module such that $R = \text{ann}(D_1) + \text{ann}(D_2)$. If D_1 and D_2 are $R-a-\delta$ -hollow modules, then M is $R-a-\delta$ -hollow module.



4. R-annihilator- δ -lifting module:

Remark: A module is R - a - δ -lifting if and only if for any c-singular submodule N of M there exists submodules K, L of M such that $M = K \oplus L$ and $N \cap K \ll_{a\delta} N$ with $\text{ann}(N) = 0$.

Proposition: 4.1 Let $M = H_1 \oplus H_2$ be duo-module. If H_1 and H_2 are R - a - δ -lifting modules, then M is R - a - δ -lifting module.

Proof: Let H_1 and H_2 are R - a - δ -lifting modules, let N be submodule of M , then $N = (N \cap H_1) \oplus (N \cap H_2)$ for each $i = \{1, 2\}$ there exists a direct summand D_i of H_i in M , such that $H_i = D_i \oplus L_i$ with $D_i \leq N \cap H_i$ and $N \cap L_i \ll_{a\delta} L_i$ with $\text{ann}(N) = 0$. then $M = (D_1 \oplus L_1) \oplus (D_2 \oplus L_2) = (D_1 \oplus D_2) \oplus (L_1 \oplus L_2)$, we have $D_1 \oplus D_2 \leq N$ and $N \cap (L_1 \oplus L_2) \ll_{a\delta} (L_1 \oplus L_2)$ with $\text{ann}(N) = 0$. Then M is R - a - δ -lifting module.

Corollary: 4.2 Let $M = H_1 \oplus H_2$ be an R -module such that If H_1 and H_2 are R - a - δ -lifting modules, then M is R - a - δ -lifting module.

A Module M is a multiplication module, if for submodule $N \subseteq M$ there exists an ideal I of R such that $N = IM$. Or $N = [N : M]M$ where $[N : M] = \{r \in R : rM \subseteq N\}$ (the annihilator of M/N). The product of two submodules N and K is well-defined as $NK = [N : M][K : M]M$ which implies to $NK = I_1 I_2 M$ if $N = I_1 M$ and $K = I_2 M$. example If R is a ring then R is itself is a multiplication module because any left ideal I is just IM ($I = IM$ for $m = R$)

Proposition: 4.3 Let M be a multiplication module. If M is R - a - δ -lifting module, then R is R - a - δ -lifting module.

Proof: [10] Assume that M is R - a - δ -lifting module. Let I be an ideal of R . Then $N = IM$ is submodule of M , hence there exists a submodules K and K' of M with $K \subseteq N$, $M = K \oplus K'$ and $(N \cap K') \ll_{a\delta} M$, with $\text{ann}(N) = 0$. But M is a multiplication module, so there are two ideals J and J' of R such that $KK' = JJ'M$ and $K = JM$ and $K' = J'M$. We get $J \subseteq I$ (since $K \subseteq N$) We have $M = K \oplus K' = JM \oplus J'M = (J + J')M$ implies that $R = J + J'$. Now $N \cap K' = IM \cap J'M \ll_{a\delta} M$ and since $(J + J')M \subseteq IM \cap J'M$, it follows that $(J \cap J') \ll_{a\delta} M$.

[4] We get $(J \cap J') \ll_{a\delta} R$, but $[(J \cap J')M : M] = J \cap J'$, therefore $(I \cap J') \ll_{a\delta} R$, Then R is R - a - δ -lifting module.

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