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Common Invariant Point Consequences with Jungck Contractive Situations and Cyclic Representations in Complete Spaces

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Abstract: The functional contractions for the cyclic maps may be seen as the functions involving multiple maps or spaces that comply with a contraction-like situation in a cyclic manner. Despite underpinning on a single distance space, a cyclical contraction is a more specialized variant, often employed in the multi-variable invariant point problems like in optimization, differential equations, or control theory. Invigorated with the work, Kirk, [12] and of other synchronous researchers/authors, we wish to propose some cyclic representations (functions), in such versions, that are other than previously proposed. These mappings may be called co-cyclic or sometimes sub-cyclic. Furthermore, introducing the notion of equivalent sequences along with some supportive as well as suitable illustrations, we state and then prove the associated common invariant point propositions, which exist uniquely, in the complete distance spaces.

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1 Introduction and Rudimental

Extending the primitive contraction notions, Kirk, *et al.*, [12] in 2003 introduced the concept of cyclic function that was further enhanced and advanced by many other field workers, can be referred to [1, 3, 5, 6, 7, 8, 11, 13, 14, 15, 16, 17, 18, 23 26]

Various authors including researchers harnessed the notion to supply many invariant point results utilizing some distinct kinds of contractions based on cyclic representation. In the year 2011, the author [26] furnished a general contractive situation for cyclic functions. In the same year, Petric [23] showed the best proximity point propositions for weak cyclic Kannan-type contractions and Karapinar with Erhan [13] did the same as well. Abbas, *et al.*, [1], Aydi, *et al.* [3], Karapinar, *et al.* and others [14, 15, 16] introduced different types of cyclic contractions and established distinct invariant point propositions.

Mathematically, a self-map (or function), say h on metric space $(\mathcal{C}, \rho)$ defined on any two non-empty sets X, Z as, $h: Y \cup Z \to Y \cup Z$ is coined to be cyclic, if [12]

$$h(Z) \subset Y, h(Y) \subset Z.$$

If so, then there must be a value $0 < \beta < 1$ in such a way that,

$$\rho(hy, hz) \leq \beta \cdot \rho(y, z), \text{ where } y \text{ and } z \text{ lie in } Y \text{ and } Z, \text{ respectively [12].}$$

The concept has been enriched by semi-cyclic functions, cyclic representations of sets and generalized cyclic representation, in the year later. Advanced the extension of cyclical representation, Abbas, *et al*. [2] introduced the cyclical notion of two functions for more than two subsets and generalized the contraction principle.

For the two subsets of a distance space, the authors [32] used cyclic representation of a pair of maps. Furthermore, the credit of the advancement of semi-cyclic maps with their fixed-point results goes to Aydi, *et al*. [4] which has been enriched by several mathematicians [21, 29, 30, 31].

In the meantime, Jungck [10] defined a new kind of contraction for a couple of functions by introducing commuting notion of mappings in distance spaces that has been advanced and modified in several ways by research community.

Inspired by all these works, we wish to introduce co-cyclic and sub-cyclic maps which are defined on a distance metric over the union of two closed subsets. Also, we are likely to state and propose common invariant point consequences associated with cyclical arrangement of mappings by employing the contractions propounded by Jungck [10], as stated above.

So, let us have some preliminary definitions and propositions as the previous traces.

Definition1.1 ([2]). *Consider a finite arrangement of non-empty subsets,* $\{\mathcal{C}_i : i = 1,2, \dots \dots \dots n\}$ *over a set* $\mathcal{C}$ *where* n *is a positive integer. The self- mappings* h, k *on* $\mathcal{C}$ *is said to have a cyclic representation with respect to* $\mathcal{C}_i$*, if the following situations hold good,*

i. $\mathcal{C} = \bigcup_{i=1}^{n} \mathcal{C}_i$,
ii. $h(\mathcal{C}_1) \subseteq k(\mathcal{C}_2), h(\mathcal{C}_2) \subseteq k(\mathcal{C}_3), \dots \dots \dots, h(\mathcal{C}_{n-1}) \subseteq k(\mathcal{C}_n), h(\mathcal{C}_n) \subseteq k(\mathcal{C}_1)$.

Here, the collection $\mathcal{C}_i$ is noted as a cyclic (h, k) cover of $\mathcal{C}$.

Definition1.2 ([2]). *The couple* (h, k) *is the generalized cyclic contraction with respect to the finite subset collection,* $\{\mathcal{C}_i : i = 1,2, \dots \dots \dots n\}$ *on metric* $(\mathcal{C}, \rho)$ *defined as:*

$$h, k : \bigcup_{i=1}^{n} \mathcal{C}_i \to \bigcup_{i=1}^{n} \mathcal{C}_i, \text{ if}$$

i. $\bigcup_{i=1}^{n} \mathcal{C}_i$ *has* (h, k) *cyclic representation with respect to collection* $\{\mathcal{C}_i : i = 1,2, \dots, n\}$
ii. $\psi\left(\rho\big(h(y), h(z)\big)\right) \leq \psi\left(C_{k,h}(y, z)\right) - \emptyset(C_{k,h}(y, z))$,

where ψ *and* $\emptyset$*, are control functions and*

$C_{k,h}(y, z) = max.\{\rho\big(k(y), k(z)\big), \rho\big(k(y), h(y)\big), \rho k(z), h(y))$, *for* $y \in C_i$, $z \in C_{i+1}, i = 1,2, \dots n$ *and* $C_{k+1} = C_1$ [2].

Definition1.3: *The maps,* $h, k : Y \cup Z \to Y \cup Z$ *on the metric space* $(\mathcal{C}, \rho)$*, of a pair of non-empty sets are semi-cyclic if* $h(Y) \subset Z$, $k(Z) \subset Y$ [4].

1.1 Jungck Contraction and Compatible Maps

Gerald Jungck extended the previous idea of contractive situations in his work and further improvised the situation for commuting mappings that ensures the existence of a ubiquitous coincidence for certain settings. Compatibility of maps and commuting with compatibility have been the key notions in Jungck, [10].

Definition1.4 ([10]). *Suppose a pair of maps,* h, k *with same domain and codomain over the metric space* $(\mathcal{C}, \rho)$*, such that,* $h(\mathcal{C}) \subset k(\mathcal{C})$*, then* $\forall \beta \in (0,1)$ *we have the contraction:*

$$\rho(h(x), h(y)) \leq \beta \cdot \rho(k(x), k(y)) \; \forall \, x, y \in C.$$

Definition 1.5: *A pair of maps, h, k with same domain and codomain over the metric space $(\mathcal{C}, \rho)$ possesses compatibility with the existence of a sequence, $(y_n)_1^\infty$ in $\mathcal{C}$, under the criteria* [10]:

$$\lim_{n\to\infty} \rho(hky_n, khy_n) = 0 \text{ with } \lim_{n\to\infty} h(y_n) = \lim_{n\to\infty} k(y_n).$$

Proposition 1.1: *A weakly commuting couple of maps are compatible but compatible pairs are not weakly commuting* [22].

Illustration 1.1 [22]. Consider the maps, $k(y) = 2y^3$ and $h(y) = y^3$, y ∈ Y. Then,

$$|k(\mathrm{y}) - h(\mathrm{y})| = |2y^3 - y^3| = |y^3| \to 0$$

$$\Leftrightarrow |kh(\mathrm{y}) - hk(\mathrm{y})| = |2(y^3)^3 - (2y^3)^3| = 6|y^9| \to 0.$$

Therefore, k and h are compatible, but not weakly commuting because, $|kh(\mathrm{y}) - hk(\mathrm{y})| \nleq |k(\mathrm{y}) - h(\mathrm{y})|\ \forall\ y \in R$ [22].

Definition 1.6: *Self-mappings h and k are compatible of type (A) if* [22],

i. $\lim_{n\to\infty} \rho\big(hk(y_n), k^2(y_n)\big) = 0,$

ii. $\lim_{n\to\infty} \rho\big(kh(y_n), h^2(y_n)\big) = 0,$

where $(y_n)_1^\infty$ is a sequence in $\mathcal{C}$, such that, $\lim_{n\to\infty} k(y_n) = \lim_{n\to\infty} h(y_n) = \alpha$, for some α.

2 Proposed Consequences

Before the final submission, first we go through the properties of new variant of a pair of sequences pronounced as equivalent and denoted as, $(x_n)_1^\infty \sim (y_n)_1^\infty$ if the terms of the sequences approach each other with the change in number of respective terms.

Definition 2.1: *The sequences $(x_n)_1^\infty$ and $(y_n)_1^\infty$ in the space $(\mathcal{C}, \rho)$ are equivalent if,*

$$\lim_{n\to\infty} \left(\frac{x_n}{y_n}\right) = 1, \text{ or } \lim_{n\to\infty} \rho(x_n, y_n) = 0.$$

Definition 2.2: *Let the sequences $(x_n)_1^\infty$ and $(y_n)_1^\infty$ are in the space $(\mathcal{C}, \rho)$ with $\lim_{n\to\infty} \left(\frac{x_n}{y_n}\right) = 1$, or $\lim_{n\to\infty} \rho(x_n, y_n) = 0$, then they converge to a common point, if they are Cauchy type and if they are not Cauchy type sequences, then asymptotically equivalent.*

Proposition 2.1: *Let the sequences $(x_n)_1^\infty$ and $(y_n)_1^\infty$ are in the space $(\mathcal{C}, \rho)$ with $\lim_{n\to\infty} \left(\frac{x_n}{y_n}\right) = 1$, or $\lim_{n\to\infty} \rho(x_n, y_n) = 0$, then $(x_n)_1^\infty \to c \Leftrightarrow (y_n)_1^\infty \to c$, as $n \to \infty$, $c \in \mathcal{C}$.*

*Proof***:** As the sequences are equivalent, so we have

$$\lim_{n\to\infty} \rho(x_n, \mathrm{y}_n) = 0 \Rightarrow \exists N_1 \in \mathbb{N} \text{ so that } \rho(x_n, \mathrm{y}_n) \le \frac{\varepsilon}{2}, \varepsilon > 0 \text{ for all } n \ge N_1.$$

Further, let $(x_n)_1^\infty \to c$ as $n \to \infty$, $c \in \mathcal{C} \Rightarrow \exists N_2 \in \mathbb{N}$ so that $\rho(x_n, c) \le \frac{\varepsilon}{2}, \varepsilon > 0$ for all $n \ge N_2$.

With the property of metric space and some suitable value $N_3 = max.\{N_1, N_2\}$, we may write,

$$\rho(y_n, c) \le \rho(x_n, \mathrm{y}_n) + \rho(x_n, c) \Rightarrow \rho(y_n, c) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \forall n \ge N_3 \Rightarrow \rho(y_n, c) \le \epsilon \Rightarrow (y_n)_1^\infty \to c, \text{ as } n \to \infty.$$

We can follow the converse part in the same manner considering the sequence $(y_n)_1^\infty \to c$, as $n \to \infty \Rightarrow (x_n)_1^\infty \to c$, as $n \to \infty$. Moreover, if $(x_n)_1^\infty$ and $(y_n)_1^\infty$ lie in closed sets Y and Z, respetively with $\lim_{n\to\infty} \rho(x_n, y_n) = 0$, then

$$\lim_{n\to\infty} \rho(x_n, y_{n+1}) = 0, \text{ or } \lim_{n\to\infty} \rho(x_{n+1}, y_n) = 0.$$

Now, let us have the details about co-cyclic and sub-cyclic maps on a pair of sub-sets of a space $(\mathcal{C}, \rho)$.

Definition 2.3: *Consider the sets Y, Z and two sets of mappings $h, k: Y \cup Z \to Y \cup Z$ on $(\mathcal{C}, \rho)$ then,*

i.h, k are co-cyclic if, $h(Y) \subset k(Y) \subset Z$ and $h(Z) \subset k(Z) \subset Y$,
ii.h sub-cyclic in k if, $h(Y) \subset k(Z) \subset Z$ and $h(Z) \subset k(Y) \subset Y$.

2.1. Invariant Point Results

From here onwards, we propose the invariant point propositions postulated on the notion of co-cyclic & sub-cyclic recommendations and respective proofs along with some supportive illustrations of established outcome.

Proposition 2.1: *Consider the non-void sets Y, Z that are closed and convex subset of a distance space $(\mathcal{C}, \rho)$ and co-cyclic type maps $h, k: Y \cup Z \to Y \cup Z$, then every x lies in $h(Z) \cap h(Y)$ implies that $h(x)$ also lies in $h(Z) \cap h(Y)$.*

Proof: Every x lies in $h(Z) \cap h(Y) \Rightarrow$ x lies in $h(Y)$ and x lies in $h(Z)$, but $h(Y) \subset Z \Rightarrow h(x) \in h(Z)$. Furthermore, $h(Z) \subset Y \Rightarrow h(x) \in h(Y)$. Hence, $h(x)$ lies in $h(Z) \cap h(Y)$.

Likewise, we can show that, every x lies in $h(Z) \cap h(Y)$ implies that $k(x)$ also lies in $k(Z) \cap k(Y)$.

Proposition 2.2: *Suppose the non-void sets Y, Z as closed and convex subset of a distance space $(\mathcal{C}, \rho)$ and co-cyclic type, continuous, and compatible maps $h, k: Y \cup Z \to Y \cup Z$ such that,*

$$\rho\big(h(x), h(y)\big) \leq \beta \cdot \rho\big(k(x), k(y)\big) \forall x \in Y, y \in Z, \forall \beta \in (0,1). \quad (2.1)$$

Then, h, k possess a common invariant point which is unique in $Y \cap Z$.

Proof: As $h(Y) \subset k(Y) \subset Z \Rightarrow \exists\ x_1 \in Y$, for $x_0 \in Y$, such that, $h(x_0) = k(x_1) \Rightarrow h(x_n) = k(x_{n+1})\ \forall\ n \in \mathbb{N}$. Again, as $h(Z) \subset k(Z) \subset Y \Rightarrow \exists\ y_1 \in Y$, for $y_0 \in Y$, such that, $h(y_0) = k(y_1) \Rightarrow h(y_n) = k(y_{n+1})\ \forall\ n \in \mathbb{N}$.

Combining (1) and above result, we write for $\forall\ x_n \in Y$ and $y_n \in Z$,

$$\rho\big(k(x_{n+1}), k(y_{n+1})\big) \leq \beta \cdot \rho\big(k(x_n), k(y_n)\big), \forall \beta \in (0,1).$$

For $n = 0$,

$$\rho\big(k(x_1), k(y_1)\big) \leq \beta \cdot \rho\big(k(x_0), k(y_0)\big).$$

For $n = 1$,

$$\rho\big(k(x_2), k(y_2)\big) \leq \beta \cdot \rho\big(k(x_1), k(y_1)\big) \leq \beta \cdot \beta \cdot \rho\big(k(x_0), k(y_0)\big)$$
$$= \beta^2 \cdot \rho\big(k(x_0), k(y_0)\big) \ldots \text{ and so on.}$$

Repeating this, we have

$$\rho\big(k(x_n), k(y_n)\big) \leq \beta^n \cdot \rho\big(k(x_0), k(y_0)\big).$$

Since $\beta \in (0,1) \Rightarrow \beta^n \to 0$ as $n \to \infty$. Therefore,

$\rho(k(x_n), k(y_n))$, as $n \to \infty$. (2.2)

Further, as Y and Z are closed subset of $\mathcal{C}$, so we can easily prove the following,

$$\lim_{n\to\infty} \rho(k(x_{n+1}), k(y_n)) = 0 \quad (2.3)$$

$$\Rightarrow \rho(k(x_n), k(x_{n+1})) \leq \rho(k(x_n), k(y_n)) + \rho(k(y_n), k(x_{n+1})).$$

Combining (2.2) & (2.3) and $n \to \infty$, we obtain

$$\rho(k(x_n), k(x_{n+1})) = 0. \quad (2.4)$$

Check whether $k(x_n)$ is a Cauchy type sequence! Let $m > 0$, then

$$\rho(k(x_n), k(x_{n+m}))$$
$$\leq \rho(k(x_n), k(x_{n+1})) + \rho(k(x_{n+1}), k(x_{n+2})) + \cdots + \rho(k(x_{n+m-1}), k(x_{n+m})).$$

Using (2.4) at $n \to \infty, \rho(k(x_n), k(x_{n+m})) = 0$. So, $(k(x_n))_0^\infty$ and $(h(x_{n-1}))_0^\infty$are both Cauchy type.

Because $(\mathcal{C}, \rho)$ is complete so the sequences, $(k(x_n))_0^\infty$ and $(h(x_{n-1}))_0^\infty$ converge to $c \in \mathcal{C}$. From proposition 2.2, the sequences $(k(x_n))_0^\infty$ and $(h(x_{n-1}))_0^\infty$ also converge to $c \in \mathcal{C}$, and $c \in Y \cap W$ as Y and Z are closed subsets of $\mathcal{C}$.

To show c is the invariant point of k, consider the sequences $(k(x_n))_0^\infty$ and $(h(x_{n-1}))_0^\infty$.

$\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} k(x_n) = c$ as $h(x_n) = k(x_{n+1})$ and h, k are continuous $\Rightarrow \lim_{n\to\infty} h(k(x_n)) = h(c)$, $\lim_{n\to\infty} k(h(x_n)) = k(c)$ and $\lim_{n\to\infty} kk(x_n) = k(c)$. Compatibility of h and k implies, $\lim_{n\to\infty} \rho(h(k(x_n)), k(h(x_n))) = 0 \Rightarrow \rho(h(c), k(c)) = 0 \Rightarrow h(c) = k(c)$. To show c is common coincidence point we use (1) as, $\rho(hk(x_n), h(x_n)) \leq \beta \cdot \rho(kk(x_n), k(x_n)) \Rightarrow \rho(h(c), c) \leq \beta \cdot \rho(k(c), c)$ as $n \to \infty$. Since $h(c) = k(c)$, therefore, $(1 - \beta) \cdot \rho(k(c), c) \leq 0 \Rightarrow k(c) = c$, because $(1 - \beta) > 0 \Rightarrow h(c) = k(c) = c$. The uniqueness of c is trivial to show.

Proposition 2.3: *Consider any two non-voids, closed and convex sets Y, Z as the subset of a distance space $(\mathcal{C}, \rho)$ and continuous compatible maps $h, k: Y \cup Z \to Y \cup Z$, such that, h is sub-cyclic with respect to k and there exists $0 < \beta < 1$ satisfying*

$$\rho(h(x), h(y)) \leq \beta \cdot \rho(k(x), k(y)) \forall\, x \in Y, y \in Z. \quad (2.5)$$

Then, h, k possess a common invariant point which is unique in $Y \cap Z$.

Proof: As, h is sub-cyclic with respect to k, $h(Y) \subset k(Z) \subset Z$, therefore, $\exists\ x_1 \in Z$, for $x_0 \in Y$, such that, $h(x_0) = k(x_1)$. And $h(Z) \subset k(Y) \subset Y \Rightarrow \exists\ x_2 \in Y$, for $x_1 \in Z$, such that, $h(x_1) = k(x_2) \Rightarrow h(x_n) = k(x_{n+1})$, $\forall n \in \mathbb{N} \cup \{0\} \Rightarrow \rho(h(x_n), h(x_{n+1})) \leq \beta \cdot \rho(k(x_n), k(x_{n+1}))$ for $0 < \beta < 1$ and $x_n \in$ Y, $x_{n+1} \in Z$. Since $h(x_n) = k(x_{n+1})$, $\forall n \in \mathbb{N} \cup \{0\} \Rightarrow \rho(k(x_{n+1}), k(x_{n+2})) \leq \beta \cdot \rho(k(x_n), k(x_{n+1}))$ $\forall n \in \mathbb{N} \cup \{0\}$, $0 < \beta < 1$ and $x_n \in$ Y, $x_{n+1} \in Z$. Chose $n = 0,1,2, \ldots$, so that we have $\rho(k(x_n), k(x_{n+1})) \leq \beta^n \cdot \rho(k(x_0), k(x_1))$.

$$\text{As } \lim_{n\to\infty} \beta^n \to 0,\ \lim_{n\to\infty} \rho(k(x_n), k(x_{n+1})) = 0 \Rightarrow \lim_{n\to\infty} \rho(h(x_n), h(x_{n+1})) = 0. \quad (2.6)$$

To show $(h(x_{n-1}))_0^\infty$ is Cauchy type sequence let even and odd positive integers positive integer a and b as $a < b$ then $x_a \in Y$ and $x_b \in Z$. Now, apply (2.5) to get,

$$\rho\big(h(x_a),h(x_b)\big) \le \beta\cdot\rho\big(k(x_a),k(x_b)\big) \le \beta\big[\rho\big(k(x_a),k(x_{a+1})\big)+\rho\big(k(x_{a+1}),k(x_{a+2})\big)+\cdots+\rho\big(k(x_{b-1}),k(x_b)\big)\big].$$

With $a,b\to\infty$, (2.6) exhibits, $\rho\big(h(x_a),h(x_b)\big)=0 \Rightarrow (h(x_{n-1}))_0^\infty$ is Cauchy type. As the space concerned is complete, so $(h(x_{n-1}))_0^\infty$ has the convergence in it, i.e., $(h(x_{n-1}))_0^\infty \to c$ as $n\to\infty$. Further, if $h(x_n)\in Z$ then $h(x_{n+1})\in Y$ and $c\in Y\cap Z$ as Y,Z are closed. This shows that $(k(x_{n-1}))_0^\infty \to c$ as $n\to\infty$.

Now, $\lim_{n\to\infty} hh(x_n)=h(c)$, $\lim_{n\to\infty} hk(x_n)=h(c)$ and $\lim_{n\to\infty} kh(x_n)=k(c)$, since h and k are continuous. And, compatibility provides, $\lim_{n\to\infty}\rho(hk(x_n),kh(x_n)=0 \Rightarrow h(c)=k(c)$.

To show c is a common invariant point, suppose $x_n\in Y$, then by (2.5), we may have

$$\beta\cdot\rho\big(k(x_n),k(hx_n)\big) \ge \rho\big(h(x_n),h(hx_n)\big)$$

$$\Rightarrow \beta\cdot\rho\big(c,k(c)\big)\ge\rho\big(c,h(c)\big),\text{ as } n\to\infty \text{ and } h(c)=k(c)$$

$$\Rightarrow \rho\big(c,h(c)\big)=0 \Rightarrow h(c)=c \Rightarrow h(c)=k(c)=c.$$

Applying the method of contradiction, suppose c is not unique and another common fixed point is $c_0\in Y\cap Z$ and let $c\in Y$ and $c_0\in Z$, then by (2.5), we get.

$$\beta\cdot\rho\big(k(c),k(c_0)\big)\ge\rho\big(h(c),h(c_0)\big)\Rightarrow\beta\cdot\rho(c,c_0)\ge\rho(c,c_0)\Rightarrow\rho(c,c_0)=0\Rightarrow c=c_0.$$

This shows the uniqueness of the fixed point in h and k.

Proposition 2.4: *Consider any two non-empty, closed and convex sets Y, Z as the subset of a distance space $(\mathcal{C},\rho)$ and continuous cyclical maps $h,k\colon Y\cup Z\to Y\cup Z$ with $h(Y)\subset Z$, $h(Z)\subset Y$ and $k(Y)\subset Z$, $k(Z)\subset Y$, there exists $0<\beta<1$ satisfying*

$$\beta\cdot\rho(x,y)\ge\rho\big(h(x),k(y)\big)\forall\, x\in Y, y\in Z. \tag{2.7}$$

Then, h,k possess a common invariant point which is unique in $Y\cap Z$.

Proof: Assume the sequences $h(x_{2n})=x_{2n+1}$ and $k(x_{2n+1})=x_{2n+2}$, $n=0,1,2$ ... Replacing x by x_{2n} and y by x_{2n+1} in (2.3), we obtain,

$$\beta\cdot\rho(x_{2n},x_{2n+1})\ge\rho(h(x_{2n}),k(x_{2n+1})$$

$$\rho(x_{2n},x_{2n+1})\ge\beta\cdot\rho(x_{2n},x_{2n+1})\ge\rho(x_{2n+1},x_{2n+2}). \tag{2.8}$$

For arbitrary values, c_0, c_1 and c_2, let $h(c_0)=c_1$ and $k(c_1)=c_2$, repeating as above, we get

$$\rho(x_{2n+1},x_{2n+2})\le\rho(x_{2n},x_{2n+1})\le\rho(x_{2n-1},x_{2n})\le\cdots\le\rho(c_0,c_1).$$

So, the sequence $c_n=(\rho(x_{2n},x_{2n+1}))_0^\infty$ is decreasing for $n=0,1,2$ ... If $\lim_{n\to\infty}c_n=l>0$, then

$$\rho(x_{2n},x_{2n+1})\ge\beta\cdot\rho(x_{2n},x_{2n+1})\ge\rho(h(x_{2n}),k(x_{2n+1}))\ \{\text{From (2.8)}\}$$

$$\rho(x_{2n},x_{2n+1})\ge\rho(x_{2n+1},x_{2n+2}).$$

If $n\to\infty$, then $\rho(l)\ge\rho(l)$. This is not possible so $l=0$.

$$\lim_{n\to\infty}\rho(x_{2n},x_{2n+1})=0. \tag{2.9}$$

Suppose a positive number m such that, $\rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \rho(x_{n+2}, x_{n+3}) + \cdots + \rho\left(x_{n+(m-1)}, x_{n+m}\right) \geq \rho(x_n, x_{n+m})$, which gives $\rho(x_n, x_{n+m}) \leq 0$, as $n \to \infty$, then $\rho(x_n, x_{n+m}) = 0$. So, the sequence concerned is Cauchy type and the distance space $(\mathcal{C}, \rho)$ is complete, therefore, there must exists a $c \in C$ such that $\lim_{n\to\infty} x_n = c$. Also, the continuity of h implies, $\lim_{n\to\infty} h(x_n) = h(c)$. But $\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} x_{n+1} = c$ $\Rightarrow h(c) = c$. Specifically, for $x = x_n$ $y = c$, with (2.5), we can state,

$\rho(x_n, c) \geq \beta \cdot \rho(x_n, c) \geq \rho(h(x_n), k(c)) \Rightarrow \rho(c, c) \geq \rho(c, g(c))$ for $n \to \infty \Rightarrow \rho(c, g(c)) = 0 \Rightarrow k(c) = c$. Along with compactness of Y and Z in $\mathcal{C}$, we say $c \in Y \cap Z$ is a common invariant point of h and k.

To follow the uniqueness of invariant point, consider a point $c_0 \in Y \cap Z$ with $h(c_0) = k(c_0) = c_0$. So, we get, $\rho(h(c_0), k(c)) \leq \beta\, \rho(c_0, c) \leq \rho(c_0, c) \Rightarrow \rho(c_0, c) \leq \rho(c_0, c) \quad \Rightarrow \rho(c_0, c) = 0 \Rightarrow c_0 = c$. This shows the uniqueness of $c \in C$.

Illustration Based on Proposition 2.2: On a complete real metric space $(\mathcal{C}, \rho)$, consider the maps $h, k: Y \cup Z \to Y \cup Z$, such that, $h(x) = -0.25x$ and $k(x) = -0.50x$ on the subsets of $\mathcal{C}$ as $Y = [-1, 0]$ and $Z = [0,1]$. Clearly, $h(Y) \subset k(Y) \subset Z$ $h(Z) \subset k(Z) \subset Y$ and h, k are compatible with respect to any sequence in $\mathcal{C}$. For $\beta = 0.50 \in (0,1)$, we have

$$\rho\left(h(x), h(y)\right) \leq \beta \cdot \rho\left(k(x), k(y)\right) \forall\, x \in Y, y \in Z.$$

Therefore, from Proposition 2.1.2, h and k must have a distinct common invariant point in $Y \cap Z$.

We can show several such illustrations even on partitions on the subsets of aforementioned subsets of $\mathcal{C}$.

3 Conclusion and Inference

This work is an analytical and conceptual extension of common fixed-point notion based on cyclic and sub cyclical functions and would extend the groundbreaking work of authors in reference [12], because if we choose identical maps ($h = k$) in Proposition 2.2 and consider the mapping k in Proposition 2.3 then both the proposed statements follow the contributing statement of the authors [12]. In this way proposing the notion we not only enrich the idea of invariant points but also generalize the result of reference [12] which may be used as the practical tool or technique in numerous practical fields, particularly where iterative methods and convergence to a unique solution are important, such as optimization, numerical methods, game theory, control theory, and engineering, among others.

4 Declarations

The author(s) declared that there is no conflict of interests.

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