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A Note on Outer Connected Accurate Domination Number of a Graph

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Abstract: An outer connected dominating set D of G is called an outer connected accurate dominating set if $V\setminus D$ has no outer connected accurate dominating set of cardinality $|D|$. The outer connected accurate domination number $\gamma_{oca}(G)$ of G is the minimum cardinality of an outer connected accurate dominating set. The outer connected accurate domination number some standard graphs are determined. .Some general properties satisfied by this concept are studied. Connected graphs of order $n \geq 2$ with outer connected accurate domination number n or $n-1$ is characterized. It is shown that for every pair of integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $\gamma_a(G) = a$ and $\gamma_{aoc}(G) = b$ where $\gamma_a(G)$ is the accurate domination number of the graph.

Keywords: outer connected accurate connected domination number, accurate domination number, outer connected domination number, dominating number.

AMS Subject Classification: 05C69.

1 Introduction

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The $order\ and\ size$ of G are denoted by $n\ and\ m$ respectively. For basic graph theoretic terminology, we refer to [1]. Two vertices $u\ and\ v$ are said to be $adjacent$ if uv is an edge of G. Two edges of G are said to be $adjacent$ if they have a common vertex. Let $S \subset V$ be any subset of vertices of G. Then the induced subgraph $G[S]$ is the graph whose vertex set is S and whose edge set consists of all of the edges in E that have both end points in S.

A subset $D \subseteq V(G)$ is called a *dominating set* if every vertex in $V \setminus D$ is adjacent to at least one vertex of D. The *domination number*, $\gamma(G)$, of a graph G denotes the minimum cardinality of such dominating sets of G. A minimum dominating set of a graph G is hence often called as a γ-set of G. The domination concept was studied in [2]. A dominating set of a graph G is a subset $D \subseteq V$ such that every vertex not in D is adjacent to at least one vertex in D. The cardinality of a smallest dominating set of G, denoted by $\gamma(G)$, is the domination number of G. The accurate domination number of G, denoted by $\gamma_a(G)$, is the cardinality of a smallest set D that is a dominating set of G and no $|D|$-element subset of $V \setminus D$ is a dominating set of G. The accurate domination concept were studied in [5]. An outer connected dominating set D is dominating and the graph $G[V \setminus D]$ is connected. The outer connected domination number of a graph G denoted by $\gamma_{oc}(G)$, is the cardinality of a minimum outer connected dominating set of G. The outer connected domination concept were studied in [4,5]. Different types of domination in graphs are studied in [6,7,8.9].

2. The outer connected accurate domination number of a graph

Definition 2.1. An outer connected dominating set D of G is called an outer connected accurate dominating set if $V \setminus S$ has no outer connected accurate dominating set of cardinality $|D|$. The outer connected accurate connected domination number $\gamma_{oca}(G)$ of G is the minimum cardinality of an outer connected accurate dominating set. Any outer connected accurate dominating set of order $\gamma_{oca}(G)$ is called a γ_{aoc}-set of G.

Example 2.2. For the graph G given in Figure 2.1, $D_1 = \{v_1, v_4\}$ is a γ_{oc}-set of G so that $\gamma_{oc}(G) = 2$. Also $D_1 = \{v_1, v_2, v_3\}$ is a γ_{oca}-set of G so that $\gamma_{oca}(G) = 3$.

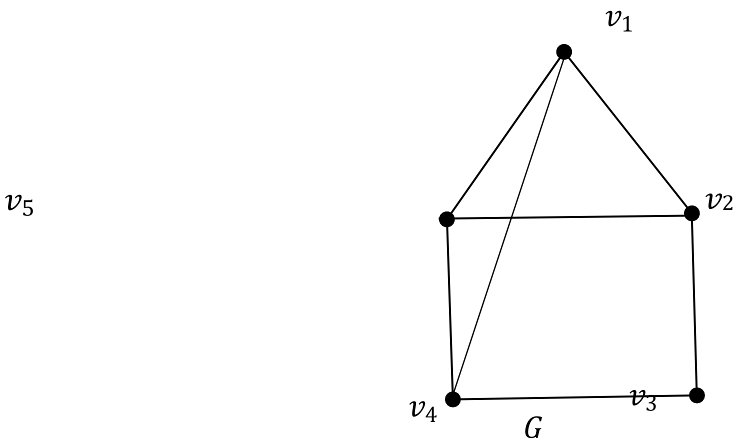

Figure 2.1

Remark 2.3. There can be more than one γ_{aoc}-set of G. For the graph G given in Figure 2.1, D_3 = $\{v_1, v_4, v_5\}$ is another γ_{oca}-set of G.

Remark 2.4. For the graph G given in Figure 2.2, $D_1 = \{v_2, v_6, v_4\}$ is a γ_a-set of G and $D_2 = \{v_1, v_2, v_5, v_7\}$ is a γ_{oca}-set of G. Therefore $\gamma_a(G)$ and $\gamma_{oca}(G)$ are different.

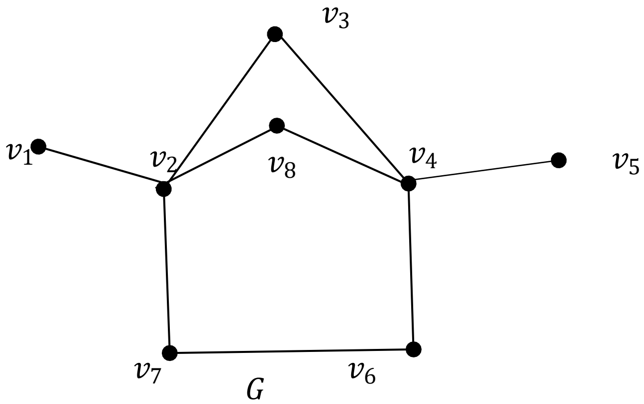

Figure 2.2

Theorem 2.5. For a star graph $G = K_{1,n-1}$ $(n \geq 3)$, $\gamma_{oca}(G) = n - 1$.

Proof. Let x be the central vertex of G and $W = \{v_1, v_2, \dots, v_{n-1}\}$ be the cut vertices of G. Then D is an outer connected accurate dominating set of G and so $\gamma_{oca}(G) \leq n - 1$. We prove that $\gamma_{oca}(G) = n - 1$. On the contrary suppose that $\gamma_{aoc}(G) \leq n - 2$. Then there exists a γ_{aoc}-set D' of G such that $|D'| \leq n - 2$. Since $V|D'$ is connected, $x \notin D'$. Let v_j $(1 \leq i \leq n - 1)$ be a vertex of G such that $v_j \notin D'$. Then v_j is not dominated by any element of D', which is a contradiction to D' a γ_{oca}-set of G. Therefore $\gamma_{aoc}(G) = n - 1$. ■ **Theorem 2.5.** For the path $G = P_n$ $(n \geq 3)$,

$$\gamma_{oca}(G) = \begin{cases} n - 1 & \text{if } n = 3 \text{ or } 4 \\ n - 2 & \text{if } n \geq 5 \end{cases}.$$

Proof. If $n = 3$ or 4, then $D = V - \{v\}$ is γ_{oca}-set of G so that $\gamma_{oca}(G) = n - 1$, where v is an end vertex of G. So let $n \geq 4$.

Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$. Let xy be an internal edge of P_n. Then $D = V(G) - \{x, y\}$ is an outer connected accurate dominating set of G and so $\gamma_{oca}(G) \leq n - 2$. We prove that $\gamma_{oca}(G) = n - 2$. On the contrary suppose that $\gamma_{aoc}(G) \leq n - 3$. Then there exists an outer connected accurate dominating set D' of G such that $|D'| \leq n - 3$. Hence it follows that $G[D']$ is a path. Then there exists at least three elements x, y and z such that $x, y, z \notin D'$. Without loss of generality, let us assume that $yz \in E(G)$. Then either y or z is not dominated by any element of D', which is a contradiction to D' a γ_{oca}-set of G. Therefore $\gamma_{oca}(G) = n - 2$. ■

Theorem 2.6. For a Bistar $B_{2,r,s}$ $(2 \le r \le s)$, $\gamma_{oca}(G) = r + s$.

Proof. Let $\{x, y\}$ be the set of central vertices of G and let $D = \{u_1, u_2, \dots, u_r\} \cup \{v_1, v_2, \dots, v_s\}$ is the set of end vertices of G, where $u_i x, v_j y \in E(G)$ $(1 \le i \le r)$ and $(1 \le j \le s)$. Then D is an outer connected accurate dominating set of G so that $\gamma_{aoc}(G) \le r + s$. We prove that $\gamma_{oca}(G) = r + s$. On the contrary suppose that $\gamma_{aoc}(G) \le r + s - 1$. Then there exists an outer connected accurate dominating set D' of G such that $|D'| \le r + s - 1$. Then $x, y \notin D'$. Hence it follows that either $u_i \notin D'$ for some $v_j \notin D'$ for some j. If $u_i \notin D'$, then u_i is not dominated by any element of D'. If $v_j \notin D'$, then v_j is not dominated by any element of D', which is a contradiction to D' a γ_{aoc}-set of G. Therefore $\gamma_{oca}(G) = r + s$. ■

Theorem 2.7. For the complete bipartite graph $G = K_{r,s}$ $(2 \le r \le s)$, $\gamma_{oca}(G) = \lfloor \frac{r}{2} \rfloor + \lfloor \frac{s}{2} \rfloor + 1$.

Proof. Let $X = \{x_1, x_2, \dots, x_r\}$ and $Y = \{y_1, y_2, \dots, y_s\}$ be the bipartite sets of G. Then $D = \{x_1, x_2, \dots, x_{\lfloor \frac{r}{2} \rfloor}\} \cup \{y_1, y_2, \dots, y_{\lfloor \frac{s}{2} \rfloor}\}$ is an outer connected accurate dominating set of G so that $\gamma_{aoc}(G) \le \lfloor \frac{r}{2} \rfloor + \lfloor \frac{s}{2} \rfloor + 1$. We prove that $\gamma_{aoc}(G) = \lfloor \frac{r}{2} \rfloor + \lfloor \frac{s}{2} \rfloor + 1$. On the contrary suppose that $\gamma_{oca}(G) \le \lfloor \frac{r}{2} \rfloor + \lfloor \frac{s}{2} \rfloor$. Then there exists a γ_{oca}-set D' with $\lfloor \frac{r}{2} \rfloor + \lfloor \frac{s}{2} \rfloor$. Then there exists at least one γ_{aoc}-set $D'' \subseteq V \setminus D'$ such that $|D''| = |D'|$, which is a contradiction to D' a γ_{aoc}-set of G. Therefore $\gamma_{oca}(G) = \lfloor \frac{r}{2} \rfloor + \lfloor \frac{s}{2} \rfloor + 1$. ■

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Theorem 2.8 For the cycle $G = C_n$ $(n \geq 4)$, $\gamma_{oca}(G) = n - 2$.

Proof. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$. Let xy be an edge of C_n. Then $D = V(G) - \{x, y\}$ is an outer connected accurate dominating set of G and so $\gamma_{oca}(G) \leq n - 2$. We prove that $\gamma_{aoc}(G) = n - 2$. On the contrary suppose that $\gamma_{aoc}(G) \leq n - 3$. Then there exists an outer connected accurate dominating set D' of G such that $|D'| \leq n - 3$. Hence it follows that $G[D']$ is a path. Then there exists at least three elements x, y and z such that $x, y, z \notin D'$. Without loss of generality, let us assume that $yz \in E(G)$. Then either y or z is not dominated by any element of D', which is a contradiction to D' a γ_{oca}-set of G. Therefore $\gamma_{oca}(G) = n - 2$. ■

Theorem 2.9. For the complete graph $G = K_n$ $(n \geq 2)$,

$$\gamma_{oca}(G) = \begin{cases} n & if\ n = 2 \\ n - 1 & if\ n = 3 \\ \frac{n}{2} + 1 & if\ n\ is\ even \\ \frac{n+1}{2} & if\ n\ is\ odd \end{cases}$$

Proof. If $n = 2$, then $D = V$ is a unique γ_{oca}-set of G so that $\gamma_{oca}(G) = n$. If $n = 3$, then $D = V - \{v\}$ is γ_{oca}-set of G so that $\gamma_{oca}(G) = n - 1$. So let $n \geq 4$. e have the following to cases.

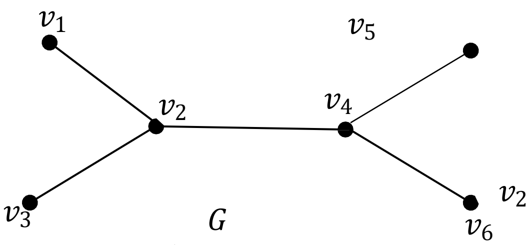
Case (i) n is even. Let $n = 2k$ $(k \geq 2)$. Consider a set S of $k + 1$ vertices of G. Then S is an outer connected accurate dominating set of G and so $\gamma_{oca}(G) \leq k + 1$. We prove that $\gamma_{oca}(G) = k + 1$. On the contrary suppose that $\gamma_{aoc}(G) \leq k$. Then there exists an outer connected dominating accurate set D' of G such that $|D'| \leq k$. Hence it follows $V \setminus D$ contains a outer connected dominating accurate set D'' of G such that $|D'| = |D''|$, which is a contradiction. Therefore $\gamma_{oca}(G) = k + 1 = \frac{n}{2} + 1$

Case (ii) n is odd. Let $n = 2k + 1$ $(k \geq 2)$. Then proceeding as in Case (i). We can prove that $\gamma_{oca}(G) = \frac{n+1}{2}$. ■

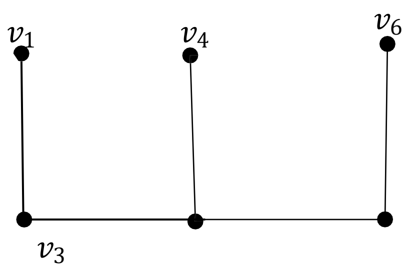
Theorem 2.10. Let T be a trivial tree of order $n \geq 2$. Then $\gamma_{oca}(G) \geq n - k$, where k is the number of vertices adjacent to the end vertices of T.

Proof. If T is K_2, then the result is obvious. So assume that $n \geq 3$. Let D be a γ_{aoc}-set of G. Then $V \backslash D$ is the set of cut vertices of T. Let $S = |V \backslash D|$. If $u \in S$, then $D = V \backslash S$ so that $\gamma_{aoc}(G) = n - k$. If $u \notin S$, then there exists a cut vertices $v \in D$ such that v is adjacent to u. Further, all vertices which are connected to u not through u also belonging to D. This implies $V \backslash D$ has at most k vertices. Hence it follows that D has at least $n - k$ vertices. Therefore γ_{oca} $(G) \geq n - k$. ■

Remark 2.11. The bound in Theorem 2.10 is sharp. For the graph G given in Figure 2.3, $D = \{v_1, v_3, v_5, v_6\}$ is a γ_{aoc}-set of G so that $\gamma_{oca}(G) = 4$. Here $n = 6$ and $k = 2$. Thus $\gamma_{oca}(G) = n - k$. The bound in Theorem 2.10 is strict. For the graph G given in Figure 2.4, $D = \{v_1, v_2, v_4, v_6\}$ is a γ_{oca}-set of G so that $\gamma_{oca}(G) = 4$. Here $n = 6$ and $k = 3$. Thus $\gamma_{oca}(G) > n - k$.



G
Figure 2.3



G
Figure 2.4

v_5

Theorem 2.12. Let G be a connected graph of order $n \geq 3$. Then $\gamma_{oca}(G) \leq n-1$. **Proof.** Let v be a vertex of G such that v is not an end vertex of G. Then $D = V - \{v\}$ is an outer connected accurate dominating set of G so that $\gamma_{oca}(G) \leq n-1$. ■

Remark 2.13. The bound in Theorem 2.12 is sharp. For the star $G = K_{1,n-1}$, $\gamma_{oca}(G) = n-1$. Also the bound in Theorem 2.12 is strict. For the graph G given in the path $G = P_n$ $(n \geq 5)$, $\gamma_{aoc}(G) = n-2$. Thus $\gamma_{oca}(G) < n-1$.

Theorem 2.14. Let G be a connected graph of order $n \geq 4$ which is neither $K_{1,n-1}$ $(n \geq 3)$ or P_4 or C_4. Then $\gamma_{oca}(G) \leq n-2$

Proof. Since $G \neq K_{1,n-1}$ $(n \geq 3)$, G contains an edge, say xy which is not an end edge of G. Let $u, v \in V$ such that $ux, vy \in E(G)$, where $u \neq v$. Then $D = V - \{u, v\}$ is an outer connected dominating set of G. Since G is neither P_4 or C_4, D is an outer connected accurate dominating set of G.so that $\gamma_{oca}(G) \leq n-2$.

Remark 2.15. The bound in Theorem 2.14 is sharp. For the graph G given in the path $G = P_n$ $(n \geq 5)$, $\gamma_{aoc}(G) = n-2$. Also the bound in Theorem 2.12 is strict. For the graph G given in Figure 2.2, $\gamma_{aoc}(G) = 4$ and $n = 7$.Thus $\gamma_{oca}(G) < n-2$.

Theorem 2.16. Let G be a connected graph of order $n \geq 3$. Then $\gamma_{oca}(G) = n-1$ if and only if G is either $K_{1,n-1}$ $(n \geq 3)$ or P_3 or P_4 or C_3or C_4.

Proof. If G is either $K_{1,n-1}$ $(n \geq 3)$ or P_3 or P_4 or C_3or C_4, then the result follows from Theorems 2.5, 2.6, 2.8, 2.9. Conversely let $\gamma_{oca}(G) = n-1$. If $n = 3$, then G is either P_3 or C_3, which satisfies the requirements of this theorem.. So, let $n \geq 4$. If G is neither $K_{1,n-1}$ $(n \geq 3)$ or P_4 or C_4, then By Theorem 2.14, $\gamma_{oca}(G) \leq n-2$, Which is a contradiction. Therefore G is either $K_{1,n-1}$ $(n \geq 3)$ or P_4 or C_4. ■

Theorem 2.17. Let G be a connected graph of order $n \geq 2$. Then $\gamma_{oca}(G) = n$ if and only if $G = K_2$.

Proof. If $G = K_2$, then the result follows from Theorem 2.9. Conversely let $\gamma_{oca}(G) = n$. If $n = 2$, then $G = K_2$, which satisfies the requirements of this theorem... If $n \geq 3$, then By Theorem 2.12, $\gamma_{oca}(G) \leq n-1$, Which is a contradiction. Therefore $G = K_2$. ■

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