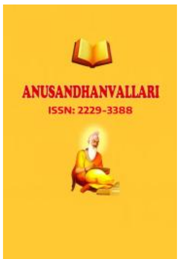

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A Study on Hyperconnectedness in *N*-Topological Spaces.

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Abstract: In this paper, the notions of $N\tau$ –hyperconnectedness, and $N\tau$ –hyperconnected components have been studied in *N-topological* spaces, and also extend it to *Nτ-pointwise* hyperconnectedness in the space. The relation between $N\tau$–hyper connected spaces and $N\tau$-pointwise hyperconnectedness has also been examined. With the help of $N\tau$ –hyperconnected components, the dimension of a N-topological space has been obtained, and also a new space, namely $N\tau$-noetherian space have been studied as an application to $N\tau$–hyperconnectedness and $N\tau$–hyperconnected components.

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1 Introduction

J.C.Kelly in 1963 [3] introduced the concept of bitopological spaces in which a non-empty set X is endowed with two arbitrary topologies. Later on, many researchers introduced and developed various forms of open sets in this space, namely $\tau_1\tau_2$[4], $\tau_{1,2}$[3]. In 2016, Lellis Thivagar et al[4] extended the concept of bitopological spaces to N topological spaces, in which a non-empty set X can be endowed with N arbitrary topologies. In this paper, we study the concept of $N\tau$ –hyperconnectedness, $N\tau$ –hyperconnected com- ponents in N-topological spaces, and also extend it to $N\tau$–point wise hyperconnectedness in the space. The relation between $N\tau$ –hyper connected spaces and $N\tau$ –point wise hyper connectedness has also been examined. Also, we discuss several characterisations and properties of *Nτ–hyperconnectedness* with suitable examples. With the help of $N\tau$ – hyper connected components, the dimension of a N-topological space has been obtained, and also a new space, namely $N\tau$-noetherian space have been studied as an application to $N\tau$–hyperconnectedness and $N\tau$–hyperconnected components.

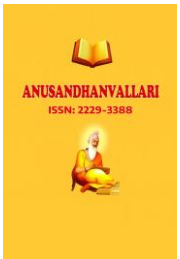
Preliminaries

In this section we have some basic definitions and concepts that will be helpful for a better understanding of this work.

Definition 2.1 [1] A quasi-pseudo metric on a non-empty set X is a function d_1 : $X \times X \rightarrow \mathrm{R}^+ \cup \{0\}$ such that

(i) $d_1(x, x) = 0$ for all $x \in X$.

(ii) $d_1(x, z) = d_1(x, y) + d_1(y, z)$ for all $x, y, z \in X$. where R^+ is the set of all positive real numbers.



Definition 2.2 [6] Let d_1 be a quasi-pseudo-metric on X, and let a function $d_2 : X \times X \to \mathbb{R}^+ \cup \{0\}$ be defined by $d_2(x, y) = d_1(y, x)$ for all $x, y \in X$. Trivially d_2 is a quasi-pseudo metric defined on X, and we say that d_1 and d_2 are conjugate one another.

If d_1 is a quasi-pseudo-metric on X, then $B_{d_1}(x, k_1) = \{y : d_1(x, y) < k_1\}$, the open d_1-sphere with centre x and radius $k_1 > 0$. Classically, the collection of all d_1- spheres forms a basis for a topology., The obtained topology be denoted by τ_1 and is called the quasi-pseudo-metric topology of d_1. Similarly we get a topology τ_2 for x, due to the quasi-pseudo-metric d_2.

Definition 2.3 [6] A non-empty set X equipped with two arbitrary topologies τ_1 and τ_2 is called a bitopological space and is denoted by (X, τ_1, τ_2).

Definition 2.4 [6] Let d_1 and d_2 be conjugate, quasi-pseudo-metrics on X and define a function $d_3 : X \times X \to \mathbb{R}^+ \cup \{0\}$ by

$$d(x, y) = \frac{[2d_1(y, x) + d_2(y, x)]}{3}, \forall x, y \in X.$$

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Then

(i) $d_3(x, x) = \frac{[2d_1(x,x)+d_2(x,x)]}{3} = 0$, for all $x \in X$

(ii) $d_3(x, z) = \frac{[2d_1(z,x)+d_2(z,x)]}{3} \le \frac{[2(d_1(z,y)+d_1(y,x))+d_2(z,y)+d_2(y,x))]}{3} = d_3(x, y) + d_3(y, z)$,

for all $x, y, z \in X$.

Therefore, d_3 is a quasi-pseudo-metric on X which is called a Mean Conjugate(simply write M.C) of d_1, d_2 and d_1. For each $i = 1, 2, 3$, the quasi pseudo-metric d_i gives a topology τ_i whose base is $\{B_{d_i}(x, k_i)\}$, where $\{B_{d_i}(x, k_i) = \{y : d_i(x, y) < k_i\}$. Thus we define a non-empty set X equipped with three arbitrary topologies $\tau_1, \tau_2,$ and τ_3 is called a tri topological space and is denoted by $(X, 3\tau)$ or $(X, \tau_1, \tau_2, \tau_3)$. Generally, let $d_1, d_2, \cdots d_{N-1}$ be quasi-pseudo-metrics on X., d_1 and d_2 be conjugate and $d_3, d_4, \cdots d_{N-1}$ be M.C of d_1, d_2 and d_1; d_1, d_2, d_3 and d_1;$\cdots$; $d_1, d_2, \cdots, d_{N-2}$ and d_1, respectively. Define a function $d_N : X \times X \to \mathbb{R}^+ \cup \{0\}$ by

$$d_N(x, y) = \frac{[d_1(y, x) + \sum_{i=1}^{N-1} d_i(y, x)]}{N} \; \forall x, y \in X$$

We can easily verify that d_N is a quasi-pseudo-metric on X. Also we note that for each N, $d_N(x, y) \neq d_N(y, x)$ for all $x, y \in X$ and d_N is called a Mean Conjugate(simply write M.C) of $d_1, d_2, \cdots, d_{N-1}$ and d_1. For each $i - 1, 2, \cdots, N$, the quasi-pseudo metric d_i gives a topology τ_i whose basis is $\{B_{d_i}(x, k_i)\}$, where $B_{d_i}(x, k_i) = \{y : d_i(x, y) < k_i\}$. Thus we define a non-empty set equipped with N-arbitrary topologies $\tau_1, \tau_2, \cdots, \tau_N$ is called a N-topological space and is denoted by $(X, N\tau)$ or$(X, \tau_1, \tau_2, \cdots, \tau_N)$.

Definition 2.5 [6] Let X be a non-empty set, $\tau_1, \tau_2, \cdots, \tau_N$ be N-arbitrary topologies defined on X and let the collection $N\tau$ be defined by

$$N\tau = \{S \subseteq X : S = (\cup_{i=1}^{N} A_i) \cup (\cap_{i=1}^{N} B_i), A_i, B_i \in \tau_i\}$$

satisfying the following axioms:

(i) $\emptyset, X \in N\tau$.

(ii) $\cup_{i=1}^{\infty} S_i \in N\tau$ for all $S_i \in N\tau$.

(iii) $\cap_{i=1}^{n} S_i \in N\tau$ for all $S_i \in N\tau$.

Then the pair $(X, N\tau)$ is called a N-topological space on X and the elements of the collection $N\tau$ are called $N\tau$-open sets on X. A subset A of X is said to be $N\tau$-closed on X if the complement of A is $N\tau$-open on X.

Definition 2.6 [6] Let Y be a non-empty subset of a N-topological space $(X, N\tau)$. Then the N-topology $N\tau_Y = \{O \cap Y : O \in N\tau\}$ is the subspace(or simply relative or induced) topology on Y for $N\tau$. The pair $(Y, N\tau_Y)$ is called a subspace of $(X, N\tau)$.

Theorem 2.7 [6] Let $(Y, N\tau_Y)$ be a subspace of $(X, N\tau)$ and $A \subseteq Y$. Then

(i) A is $N\tau_Y$-closed in Y if and only if $A = Y \cap F$, where F is $N\tau$-closed in X.

(ii) A is $N\tau_Y$-closed in Y and Y is $N\tau$-closed in X. Then A is $N\tau$-closed in X.

Lemma 2.8 (Zorn's Lemma)[9] Let A be a non-empty family of sets, with the property that for each chain $\{A_\alpha : \alpha \in I\}$ in A, we have that $\cup_{\alpha \in I} A_\alpha \in A$. Then there is a maximal element in A.

Definition 2.9 Let $(X, N\tau)$ be a N-topological space. If $N\tau = P(X)$, Then $N\tau$ is the discrete N-topology on X. If $N\tau = \{\emptyset, X\}$, then $N\tau$ is the indiscrete N-topology on X.

2 $N\tau$ -Hyperconnectedness

In this section we introduce the concept of $N\tau$–Hyperconnectedness in N-topological spaces and discuss some of its properties.

Definition 3.1 A N-topological space $(X, N\tau)$ is said to be $N\tau$-connected if it cannot be written as a union of two non-empty proper disjoint $N\tau$-closed subsets of X.

Definition 3.2 A N-topological space $(X, N\tau)$ is said to be $N\tau$–hyperconnected if it cannot be written as a union of two non-empty proper $N\tau$-closed subsets of X.

Example 3.3 (i) Let $X = \{a, b, c, d\}$, $\tau_1 = \{\emptyset, X, \{a, c\}, \{a, c, d\}, \tau_2 = \{\emptyset, X, \{a, b, c\}\}\}$ be two topologies on X and $2\tau = \{\emptyset, X, \{a, c\}, \{a, c, d\}, \{a, b, c\}\}$ be the corre- sponding N-topology on X. Then, $N\tau^c = \{\emptyset, X, \{b, d\}, \{b\}, \{d\}\}$ and hence X is $N\tau$–hyperconnected.

(ii) Let X be an infinite set. Let $\tau_1 = \{\emptyset, X, A\}$, where $A = \{U \subseteq X : U^c \text{ is finite}\}$, $\tau_2 = \{\emptyset, X, \}$. Then $2\tau = \{\emptyset, X, A\}$, where $A = \{U \subseteq X : U^c \text{ is finite}\}$ is the 2τ topology on X and $(X, 2\tau)$ is 2τ–hyperconnected.

Remark 3.4 A subset Y of a N-topological space $(X, N\tau)$ is N–hyperconnected if it is N–hyperconnected with respect to the subspace topology $N\tau_Y$ on Y.

Theorem 3.5 Let $(X, N\tau)$ be a N-topological space. Then the following are equivalent:

(i) X is $N\tau$–hyperconnected.

(ii) Every pair of non-empty $N\tau$-open subsets of X has a non-empty intersection.

Proof: We prove by showing that the negation of the two statements are equivalent. That is, X is not $N\tau$–hyperconnected ⇔ there exists a pair of non-empty $N\tau$-open sets which do not intersect.

X is not $N\tau$–hyperconnected ⇔ $X = A_1 \cup A_2$, where $A_1$and A_2 are non-empty proper closed subsets of X ⇔ we can find non-empty closed subsets A_1 and A_2 such that $(X \backslash A_1) \cap (X \backslash A_2) = \emptyset$ ⇔ there exists $N\tau$-open sets $X \backslash A_1$ and $X \backslash A_2$ such that $X \backslash A_1 \cap X \backslash A_2 = \emptyset$, which completes the proof.

Theorem 3.6 The property of being $N\tau$–hyperconnected is open hereditary.

Proof: Let A be a $N\tau$-open subset of $N\tau$–hyper connected space $(X, N\tau)$. We have to prove A is $N\tau_A$-hyperconnected. If $A = \emptyset$, then A is $N\tau_A$-hyperconnected, since the empty set has no non-empty proper $N\tau$ -closed sets. So now let $A \nvDash \emptyset$. Let U_1 and U_2 be two non-empty $N\tau_A$-open subsets of A. Since every non-empty $N\tau_A$-open subsets of A is a non-empty open subset of X, we have U_1 and U_2 are $N\tau$ open in X and since X is $N\tau$ –hyper connected by theorem 3.5(ii) we have, $U_1 \cap U_2 \nvDash \emptyset$ which implies A is N-hyperconnected(again by using Theorem 3.5 for A). This completes the proof.

Remark 3.7 A closed subset of a N–hyperconnected space is not N–hyperconnected as shown by the following example.

Example 3.8 Let $X = \{a, b, c\}, \tau_1 = \{\emptyset, X, \{a\}, \{a, c\}\}, \tau_2 = \{\emptyset, X, c\}$. Then $2\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ is the 2τ-topology on X. Let $Y = \{a, b\}$, a closed subset of X.

Then $2\tau_Y = \{\emptyset, Y, \{a\}\}$ is the 2τ-subspace topology on Y. Now $2\tau^c{}_Y = \{\emptyset, Y, \{b, c\}\}$ and hence Y is not a hyper connected subset of X.

Remark 3.9 Theorem 3.6 and remark 3.7 concludes that the property of being $N\tau$– hyperconnected is not hereditary.

Proposition 3.10 Let $(X, N\tau)$ be a N-topological space. A subset Y of X is $N\tau_Y$– hyper connected if and only if $N\tau$-cl(Y) is hyperconnected.

Proof: Proof follows directly from the fact that every $N\tau$-open set that intersects Y also intersects $N\tau$-cl(Y).

Definition 3.11 A N-topological space $(X, N\tau)$ is said to be $N\tau$-connected if it cannot be written as a union of two non-empty proper disjoint closed subsets of X.

Remark 3.12 Every $N\tau$–hyperconnected space is $N\tau$-connected. But the converse does not hold as shown by the following example.

Example 3.13 Let $X = \{a, b, c\}, \tau_1 = \{\emptyset, X, \{a\}, \{a, c\}\}, \tau_2 = \{\emptyset, X, c\}$. and $2\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ is the 2τ-topology on X and $2\tau^c = \{\emptyset, X, \{b, c\}, \{a, b\}, \{b\}\}$. Then we have $(X, 2\tau)$ is 2τ-connected but not 2τ–hyperconnected, since $X = \{a, b\} \cup \{b, c\}$ and $\{a, b\}, \{b, c\} \in 2\tau^c$.

Theorem 3.14 In a N-topological space $(X, N\tau)$, the closure of a one point set $\{x\}, x \in X$ is $N\tau_{\{x\}}$-hyperconnected.

Proof: Let $x \in X$ and $N\tau$-cl$(\{x\}) = A$. Suppose that $A = A_1 \cup A_2$, where A_1 and A_2 are non-empty proper $N\tau_A$-closed subsets of A. Since every $N\tau_A$ closed subset of a $N\tau$-closed set A is $N\tau$-closed, we have A_1 and A_2 are $N\tau$-closed subsets of X. We have $A_1 \subseteq A$ and $A_2 \subseteq A$ and x must be contained in either of A_1 or A_2, which contradicts the fact that A is the smallest closed set containing x and hence A must be N–hyperconnected. This completes the proof.

Definition 3.15 A N-topological space $(X, N\tau)$ is said to be $N\tau$-*Hausdorff* if for every $x, y \in X$ with $x \neq y$, there exists $N\tau$-open sets U and V in X such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Theorem 3.16 Let $(X, N\tau)$ be a N-topological space. If X is $N\tau$–hyperconnected and

$N\tau$-Hausdorff, then X contains at most one point.

Proof: Proof is by method of contradiction. Suppose that X consists of at least two points. Let $u, v \in X$ and $u \neq v$. Since, X is $N\tau$-Hausdorff, there exists two $N\tau$-open sets U and V containing u and v respectively such that $U \cap V = \emptyset$. Therefore, there exists two non-empty $N\tau$-open sets in X such that $U \cap V = \emptyset$, which contradicts the assumption that X is N–hyperconnected. Thus X is a one point space and this completes the proof.

Definition 3.17 Let $(X, N\tau)$ be a N-topological space. A subset $A \subseteq X$ is said to be

$N\tau$-dense in X if $N\tau$-cl$(A) = X$. Otherwise it is said to be not $N\tau$-dense.

Definition 3.18 Let $(X, N\tau)$ be a N-topological space. A subset $A \subseteq X$ is said to be nowhere $N\tau$-dense in X if $N\tau$-int$(N\tau$-cl$(A) = \emptyset$.

Theorem 3.19 Let $(X, N\tau)$ be a $N\tau$–hyperconnected space and A be a non-empty

$N\tau$-open set in X. Then A is $N\tau$-dense in X.

Proof: Let A be a non-empty $N\tau$-open subset of X. Then $(X\backslash A)$ is $N\tau$-closed in X. Also we have $X = (X\backslash A) \cup (N\tau$-cl$(A))$. Since X is $N\tau$–hyperconnected, one of these $N\tau$-closed subsets must be equal to X. Since, $A \neq \emptyset$, we have $X\backslash A \neq X$ and hence $N\tau$-cl$(A) = X$, which implies A is $N\tau$-dense in X. Thus we have every non-empty $N\tau$-open subset of a $N\tau$–hyperconnected space is $N\tau$-dense.

Theorem 3.20 Let $(X, N\tau)$ be a $N\tau$–hyperconnected Space and A be a subset of X

with $N\tau$-int$(A) \neq \emptyset$. Then A is $N\tau_A$-hyperconnected.

Proof: Let T be any non-empty $N\tau_A$-open set in A. Then $T = S \cap A$ for some non-empty $N\tau$-open set S in X. Since $N\tau$-int$(A) \neq \emptyset$, we have $S \cap N\tau$-int(A) is a non- empty $N\tau$ open set in X. Since X is $N\tau$–hyperconnected, by theorem(Above) $S \cap N\tau$-int(A) is $N\tau$-dense in X. Therefore $X = N\tau$-cl$(S \cap N\tau$-int$(A)) \subseteq N\tau$-cl$(S \cap A) = X$, which implies $N\tau$-cl$(S \cap N\tau$-int$(A)) = N\tau$-cl$(S \cap A) = X$. Thus $N\tau_A cl(T) = N\tau$- cl$(S \cap A) \cap A = X \cap A = A$, which implies T is $N\tau_A$-dense in X, and hence by theorem (3.19) A is $N\tau_A$-hyperconnected.

Theorem 3.21 A N-topological space $(X, N\tau)$ is $N\tau$–hyperconnected if and only if union of two not $N\tau$-dense set is a not $N\tau$-dense set.

Proof: Let $(X, N\tau)$ be a $N\tau$–hyperconnected space and let S and T be two not $N\tau$- dense sets in $(X, N\tau)$. Then there exists $N\tau$-open sets U and V in X such that $S \cap U = \emptyset$ and $T \cap V = \emptyset$. Since X is $N\tau$–hyperconnected, by theorem(3.5(ii)) we have $U \cap V \neq \emptyset$. Thus we have $(S \cup T) \cap (U \cap V) = \emptyset$, and hence $S \cup T$ is not $N\tau$-dense. To prove the converse part let us prove by the method of contradiction. Suppose that the converse holds and $(X, N\tau)$ is not $N\tau$–hyperconnected. Then there exists non-empty $N\tau$-open sets E and F in X such that $E \cap F = \emptyset$. Hence $E \subseteq X\backslash F$ and $F \subseteq X\backslash E$. Then we have $X\backslash E$ and $X\backslash F$ are not $N\tau$-dense in X. But we have $X\backslash E \cup X\backslash F = X$, which is a contradiction to our assumption, since X is $N\tau$ -dense in X. Hence we have X is $N\tau$ –hyperconnected and this completes the proof.

Theorem 3.22 If in a N-topological space $(X, N\tau)$ the union of two not $N\tau$ -dense sets is a not $N\tau$ -dense set, then the class of not $N\tau$ -dense sets coincides with the class of nowhere $N\tau$ -dense sets.

Proof: Let $\neg N_{0d}(N\tau)$ and $\neg N_{wd}(N\tau)$ denoted the class of not $N\tau$-dense and nowhere $N\tau$-dense subsets of $(X, N\tau)$ respectively. Clearly we have $N_{wd}(N\tau) \subseteq N_{od}(N\tau)$. Now let $A \in N_{od}(N\tau)$ and suppose $A \notin N_{wd}(N\tau)$. Then we have $N\tau$-int($N\tau$-cl(A)) $\nvDash \emptyset$, which implies there exists a $N\tau$-open set $U \in N\tau$ such that $U \subseteq N\tau$-cl(A). Then clearly we have $N\tau$ -cl(X-U) $\notin N_{od}(N\tau)$. Now $N\tau$ -cl($A \cup (X$-U))=$N\tau$ -cl(A)$\cup N\tau$ -cl(X-U) $= X$, which is a contradiction to the hypothesis that the union of two not $N\tau$ -dense sets is a not $N\tau$ -dense set, and here X is a $N\tau$ -dense set. Thus $A \in N_{wd}(N\tau)$ and hence we have $N_{0d}(N\tau) = N_{wd}(N\tau)$ and this completes the proof.

Corollary 3.23 A N-topological space $(X, N\tau)$ is $N\tau$ –hyperconnected if and only if the collection of not $N\tau$-dense sets and nowhere $N\tau$-dense sets in X are equal.

3 Hyperconnected components in $N\tau$

In this section, we discuss about $N\tau$ –hyperconnected components by virtue of $N\tau$ – hyper connectedness and study some of its properties.

Definition 4.1 Let $(X, N\tau)$ be a N-topological space. A subset $F \subset X$ is said to be maximal $N\tau F$–*hyperconnected* if

(i) F is $N\tau_F$–hyperconnected and

(ii) If there exists a $A \subseteq X$ such that A is $N\tau_A$-hyperconnected and $F \subset A$, then $F = A$.

Definition 4.2 A maximal $N\tau$ –hyperconnected subset of a N-topological space $(X, N\tau)$ is said to be a $N\tau$ -hyperconnected component of X.

Theorem 4.3 Let $(X, N\tau)$ be a N-topological space. Then,

(i) Every $N\tau$ –hyperconnected subset of X is contained in a $N\tau$ –hyperconnected com- ponent of X.

(ii) Every $N\tau$–hyperconnected component of X is $N\tau$-closed in X and they cover X.

Proof:

(i) Let A be a $N\tau_A$-hyperconnected subset of X and let S be the family consisting of all $N\tau$ –hyper connected subsets of X that contain A. Now we show that for every chain $T = \{B_\alpha\}_{\alpha\in I}$ in S we have $B = \cup_{\alpha\in I} B_\alpha$ is $N\tau_B$-hyperconnected. For let U and V be two non-empty $N\tau_B$-open sets in B. Then there exist α and β in S such that $U \cap B_\alpha$ and $V \cap B_\beta$ are non-empty, as each member of the union is N–hyper connected. Since T is a chain, we have either $B_\alpha \subseteq B_\beta$ or $B_\beta \subseteq B_\alpha$ and thus either the sets $U \cap B_\alpha$ and $V \cap B_\alpha$, or the sets $U \cap B_\beta$ and $V \cap B_\beta$, are non-empty. In particular, $(U \cap B) \cap (V \cap B)$ is non-empty, and thus by theorem 3.5(ii) we have B is $N\tau_B$-hyperconnected. Since union is clearly an upper bound for a chain, we get a maximal $N\tau$ –hyperconnected element for every chain T in S, and hence by Zorn's Lemma, S have a maximal $N\tau$ –hyperconnected element containing A, and this completes the proof.

(ii) Proof of (ii) follows directly from Theorem(3.10) and (i).

Remark 4.4 The N–hyperconnected component obtained in Theorem(4.3(i)) is not unique, as shown by the following example.

Example 4.5 Let $X = \{1, 2, 3\}$, $\tau_1 = \{\emptyset, X, \{2\}, \{2, 3\}\}$, $\tau_2 = \{\emptyset, X, \{3\}\}$. Then, $2\tau =$

$\{\emptyset, X, \{2\}, \{3\}, \{2, 3\}\}$. Then we have $A = \{1\}$ is $2\tau_A$-hyperconnected and it is contained in $\{1, 3\}$ and $\{1, 2\}$ which are 2τ–hyperconnected components of X.

4 Pointwise hyperconnectedness in N-topological spaces

In this section, we extend the concept of $N\tau$ –hyperconnectedness to pointwise $N\tau$ - hyperconnectedness and study its relation in the space.

Definition 5.1 A N-topological space $(X, N\tau)$ is said to be point wise $N\tau$–hyper connected at a point $x \in X$ if each $N\tau$-open set containing x is $N\tau$-dense in X.

Example 5.2 Let $X = \{a, b, c\}\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$, $\tau_2 = \{\emptyset, X, \{b\}\}$, $\tau_3 = \{\emptyset, X, \{a, b\}\}$. Then, $3\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ is a tri-topology on X. Then we have X is pointwise 3τ–hyperconnectected at c but neither at a nor b.

Remark 5.3 A N-topological space $(X, N\tau)$ is $N\tau$–hyperconnected if and only if it is pointwise $N\tau$ –hyperconnected at each of its points.

Theorem 5.4 Let $X, N\tau$ be a N-topological space, then the set of all points at which

X is pointwise $N\tau$–hyper connected is a $N\tau$-closed subset of X.

Proof: Let P denote the set of all pointwise $N\tau$ –hyperconnected members of X. To prove P is $N\tau$ -closed let us prove that X-P is $N\tau$ -open. Let $x \in X$-P . Then by the definition of pointwise $N\tau$ –hyper connectedness, there exists a $N\tau$ -open set U containing x and $N\tau$ -cl$(U) \nvDash X$ and thus $U \subseteq X$-P . Thus we have X-P is a union of $N\tau$ -open sets, and hence by the definition of $N\tau$ -open sets, X-F is $N\tau$ -open in X. Hence F is $N\tau$-closed in X and this completes the proof.

Theorem 5.5 In a N-topological space $(X, N\tau)$, the following are equivalent:

(i) X is pointwise $N\tau$–hyperconnected at x.

(ii) Every non-empty $N\tau$-open set intersects every $N\tau$-open set containing x.

(iii) Every $N\tau$-open set containing x is $N\tau$-connected.

(iv) Every closed subset of X not containing x is nowhere $N\tau$-dense in X.

Proof: (i)⇒(ii) and (ii) ⇒(iii) are straightforward.

(iii⇒)(iv) Let B be a non-empty $N\tau$ -closed set not containing x. Then we hace X-B is $N\tau$ -open in X. Suppose B is not nowhere $N\tau$ -dense in X. Then we have $N\tau$ -int$(B) \nvDash \emptyset$. Let $U = N\tau$ -int$(B) \cup X$-B. Then U is a $N\tau$ -open set containing x which is not $N\tau$ - connected, and hence a contradiction to our assumption. Thus we have B is nowhere

$N\tau$-dense in X.

(iv) ⇒ (i) Let U be a $N\tau$ -open set containing x. Suppose U is not $N\tau$ -dense in X. Then there exists a non-empty $N\tau$ -open set V in X such that $U \cap V = \emptyset$. Then we have X-U is a non-empty $N\tau$ -closed set not containing x and not nowhere $N\tau$ -dense in X, which is a contradiction to our assumption. Hence we have U is $N\tau$ -dense in X and thus X is pointwise $N\tau$ –hyperconnected at x.

Theorem 5.6 Let $(X, N\tau)$ and $(Y, N\tau^r)$ be two *N-topological* spaces. If $f : X \to Y$ is a N-continuous surjection and X is pointwise $N\tau$ –hyperconnected at x, then Y is pointwise $N\tau^r$-hyperconnected at $f(x)$.

Proof: Let $V \subseteq Y$ be a $N\tau$-open set containing $f(x)$. Since f is N-continuous $f^{-1}(V)$ is $N\tau$-open in X and it contains x. Since X is pointwise $N\tau$–hyperconnected at x, we have $f^{-1}(V)$ is $N\tau$-dense in X. Since f is a N-continuous surjection, it preserves $N\tau$-dense sets, and hence we have $V = f(f^{-1}(V))$ is $N\tau^r$-dense in Y. Thus Y is $N\tau^r$- hyperconnected at x and this completes the proof.

Theorem 5.7 Let $(X, N\tau)$, $(Y, N\tau^r)$ be N-topological spaces, where X is pointwise $N\tau$ – hyperconnected at $x \in X$ and Y is a $N\tau^r$-Hausdorff space. Let $f : X \to Y$ be a N-continuous injective map. Then, f is constant.

Proof: If X has only one element, the theorem is obvious. So let X have at least two elements and suppose that f is not constant. Then there exists some $y \in X$ such that $f(x) \neq f(y)$ in Y. Since, Y is $N\tau^r$-Hausdorff, there exists $N\tau^r$- open sets E and F in Y containing $f(x)$ and $f(y)$ respectively such that $E \cap F = \emptyset$. Since f is N-continuous, $f^{-1}(E)$ and $f^{-1}(F)$ are non-empty $N\tau$-open sets in X and $f^{-1}(E)$ contains x. Since f is injective we have $f^{-1}(E) \cap f^{-1}(F) = \emptyset$, which is a contradiction to Theorem 5.5(iv) . Thus f is constant and this completes the proof.

5 Applications

In this section let us deal with some applications of $N\tau$ –hyperconnectedness and $N\tau$ – hyperconnected components.

Definition 6.1 Let $(X, N\tau)$ be a N-topological space.

1. A chain of $N\tau$–hyperconnected closed subsets of X is a sequence $W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_n \subset X$ with each W_i is $N\tau$-closed hyperconnected and $W_i \neq W_{i+1}$ for $i = 1, 2, \cdots, n-1$.

2. The length of a chain $W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_n \subset X$ of $N\tau$–hyperconnected closed subsets of X is the integer n.

Definition 6.2 Let $(X, N\tau)$ be a N-topological space. The $N\tau$-dimension of X is defined to be the supremum of the length n of all the chains $W_0 \subset W_1 \subset \cdots W_n$ of $N\tau$–hyperconnected closed subsets W_i of X. We denote the $N\tau$-dimension of X by $N\tau$-dim(X).

Example 6.3 (i) Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$, $\tau_2 = \{\emptyset, X, \{b\}\}$, $\tau_3 = \{\emptyset, X, \{c\}, \{b, c\}\}$. Then $3\tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ is the 3τ-topology on

X. Then 3τ-dim$(X, 3\tau) = 0$.

(ii) Let $X = \{1, 2\}$, $\tau_1 = \{\emptyset, X\}$, $\tau_2 = \{\emptyset, X, \{1\}\}$. Then $2\tau = \{\emptyset, X, \{1\}\}$ is the 2τ-topology on X. Then 2τ-dim$((X, 2\tau))= 1$.

Definition 6.4 Let $(X, N\tau)$ be a N-topological space and Y be a $N\tau$–hyperconnected closed subset of X. Then we define the $N\tau$-codimension of Y in X to be the supremum of the length n of all chains $Y = W_0 \subset W_1 \subset \cdots W_n$ of $N\tau$–hyperconnected closed subsets W_i of X. We denote the $N\tau$-codimension of Y in X by $N\tau$-codim(Y, X).

Remark 6.5 Let $(X, N\tau)$ be a N-topological space and $\{X_\alpha\}_{\alpha \in I}$ its $N\tau$–hyperconnected components. Then $N\tau$-dim(X)= $\sup_{\alpha \in I} N\tau$-dim$(X_\alpha)$.

Definition 6.6 A N-topological space is said to be $N\tau$-Noetherian, if it satisfies the descending chain condition on the $N\tau$-closed subsets of X: for any sequence of $N\tau$ - closed sets $F_1 \supset F_2 \supset \cdots$, there exists $r \geq 1$ such that $F_i = F_r$ for all $i \geq r$,(*i.e*).,every chain of $N\tau$-closed subsets of X have a minimal element.

Example 6.7 Let $X = \{a, b, c, d\}$,and $\tau_1 = \{\emptyset, X, \{a\}, \{b, c, d\}\}$, $\tau_2 = \{\emptyset, X, \{b\}, \{a, c\}, \{a, b, c\}\}$, $\tau_3 = \{\emptyset, X, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}\}$. Then $3\tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{c, d\}, \{b, c\}, \{a, b\}, \{b, c\}, \{a, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, \{a, b, d\}\}$ is a 3τ-topology on X and $(X, 3\tau)$ is a 3τ-Noetherian space.

Theorem 6.8 A $N\tau$ -Noetherian topological space $(X, N\tau)$ has only a finite number of distinct $N\tau$ –hyperconnected components $X_1, X_2, .., X_n$. Moreover, X_i is not contained in $\cup_{i \models j} X_j$ for $i = 1, 2, ..., n$.

Proof: Let F be the collection of all $N\tau$ -closed subsets of X for which the theorem does not hold. Suppose that F is non-empty. Since X is $N\tau$ -Noetherian, the collection F has a minimal element, say E. Then E is not $N\tau$ –hyperconnected and thus $E = C \cup D$, where C and D are non-empty, proper $N\tau$ -closed subsets of E. By the minimality of E, both the closed sets C and D have a finite number of $N\tau$ –hyper connected components. Thus, E can be written as a union of a finite number of $N\tau$ -closed hyperconnected subsets of X. Hence by theorem(4.3(i) , we can conclude that E has only a finite number of $N\tau$ –hyperconnected components. This a contradiction to the assumption that F is non- empty. Hence, F is empty and thus the theorem holds. Further if there exists an i such that $X_i \subseteq \cup_{i \models j} X_j$, we have X_i is covered by the $N\tau$ -closed subsets $X_i \cap X_j$ for some $i \models j$. Since X_i is $N\tau_{X_i}$ –hyper connected, we have X_i is contained in one of the X_j, which contradicts the maximality of X_i. Thus there is no i such that $X_i \subseteq \cup_{i \models j} X_j$. This completes the proof.

6 Conclusion

The notions of $N\tau$ –hyperconnectedness, $N\tau$ –hyperconnected components, pointwise $N\tau$ – hyperconnectedness in N-topological spaces have been introduced and studied. Some of its characterizations and properties are studied and obtained. As an application to the introduced concepts, the idea of dimension and co-dimension of an N-topological space has been obtained, also a new space namely $N\tau$ -Noetherian space also have been introduced. The introduced concepts can be further extended to bear many more inter- esting results and also can be extended to study weak and stong form of open sets in this space.

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